

Finding Functions of Time

Some forces are not constant, but because they vary in some regular way, they can be modeled as a function of time (such as $F = 3t^2 + 2$) or as a function of the object's velocity (such as $F = kv^2$). The drag force (such as air resistance) is an example of a force that is a function of velocity.

The purpose of this assignment is to learn how to start with a force that is a function of time or velocity (such as $F = kt^2$ or $F = -kv$) and derive expressions for other quantities as functions of time.

If a derivation is provided, write down what was done in each step. Then cover it and do the derivation on your own in the space to the right. Since we will only be dealing with objects moving in one dimension, we will simplify the notation by dropping the subscripts that indicate x or y components, and write a instead of a_x or a_y , and v instead of v_x or v_y .

1. Given a force as a function of time, $F(t)$

An object of mass 2 kg is moving along the x-axis. It experiences a single horizontal force that varies with time as $F(t) = 2t - 4t^2$, where time is in seconds and force in newtons. At $t = 0$, the particle was at $x = 3$ m and its horizontal velocity was 5 m/s.

a. Find the object's acceleration as a function of time.

$$a_x = \frac{\Sigma F_x}{m} \quad \text{write N2L}$$

$$a = \frac{2t - 4t^2}{2\text{kg}} \quad \text{put in } F, m$$

$$a = t - 2t^2 \quad \text{divide by 2}$$

(your derivation here!)

b. Find the object's velocity as a function of time.

$$a = t - 2t^2 \quad \text{write } a(t) \text{ function}$$

$$\frac{dv}{dt} = t - 2t^2 \quad \text{replace } a \text{ with } \frac{dv}{dt}$$

$$dv = (t - 2t^2) dt \quad \text{multiply both sides by } dt$$

$$\int dv = \int (t - 2t^2) dt \quad \text{take integral of both sides}$$

$$v = \frac{1}{2}t^2 - \frac{2}{3}t^3 + C \quad \text{integrate!}$$

$$\left[\begin{array}{l} 5 \text{ m/s} = \frac{1}{2}(0)^2 - \frac{2}{3}(0)^3 + C \\ 5 \text{ m/s} = C \end{array} \right] \quad \text{evaluate } C \text{ using } t=0 \text{ info.}$$

$$v = \frac{1}{2}t^2 - \frac{2}{3}t^3 + 5 \text{ m/s} \quad \text{write } v(t) \text{ with } C \text{ substituted}$$

(your derivation here!)

c. Find the object's position x as a function of time:

$$v = \frac{1}{2}t^2 - \frac{2}{3}t^3 + 5$$

$$\frac{dx}{dt} = \frac{1}{2}t^2 - \frac{2}{3}t^3 + 5$$

$$\int dx = \int \left(\frac{1}{2}t^2 - \frac{2}{3}t^3 + 5 \right) dt$$

$$x = \frac{1}{2} \cdot \frac{1}{3} t^3 - \frac{2}{3} \cdot \frac{1}{4} t^4 + 5t + C$$

$$x = \frac{1}{6} t^3 - \frac{1}{6} t^4 + 5t + C$$

$$\left[\begin{array}{l} 3\text{m} = 0 - 0 + 0 + C \\ 3\text{m} = C \end{array} \right]$$

$$x = \frac{1}{6} t^3 - \frac{1}{6} t^4 + 5t + 3 \text{ m}$$

d. Find the object's kinetic energy as a function of time: \pm is not necessary to simplify

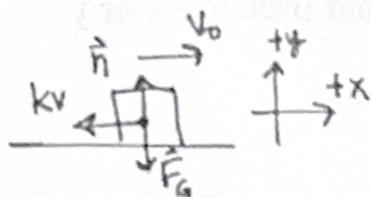
$$K = \frac{1}{2}mv^2 = \frac{1}{2}(2\text{kg})\left(\frac{1}{2}t^2 - \frac{2}{3}t^3 + 5\right)^2$$

$$K = \left(\frac{1}{2}t^2 - \frac{2}{3}t^3 + 5\right)^2$$

2. Given a force as a function of velocity, $F(v)$

A block of mass m , which was at $x = 0$ and moving with velocity v_0 at $t = 0$, experiences a single horizontal force that is opposite its motion and has magnitude $F = kv$, where k is a constant, velocity is in m/s, and force is in newtons.

a. Find the block's velocity as a function of time. (It is always necessary to find velocity as a function of time before you can find position, acceleration, or force as functions of time.)



Sketch, force diagram, coordinate system

$$a_x = \frac{\sum F_x}{m}$$

write N2L

(your derivation here!)

$$a = -\frac{kv}{m}$$

put in $-kv$ for $\sum F_x$

$$\frac{dv}{dt} = -\frac{kv}{m}$$

replace a with $\frac{dv}{dt}$

$$dv = \left(-\frac{kv}{m}\right)dt$$

multiply both sides by dt

$$\frac{dv}{v} = -\frac{k}{m} dt$$

divide both sides by v

$$\int \frac{dv}{v} = \int -\frac{k}{m} dt$$

because all the velocity things need to be on same side for integration
take integral of both sides

$$\int v^{-1} dv = -\frac{k}{m} \int dt$$

change $\frac{1}{v}$ to v^{-1} , put constants out front

$$\ln v = -\frac{k}{m}t + C$$

do the integration

$$\left[\begin{aligned} \ln v_0 &= -\frac{k}{m}(0) + C \\ \ln v_0 &= C \end{aligned} \right]$$

evaluate the constant using $v = v_0$ at $t = 0$.

$$\ln v = -\frac{k}{m}t + \ln v_0$$

put in the constant C

$$\ln v - \ln v_0 = -\frac{k}{m}t$$

move $\ln v_0$ to left side

$$\ln\left(\frac{v}{v_0}\right) = -\frac{k}{m}t$$

re-write $\ln v - \ln v_0$ as $\ln\left(\frac{v}{v_0}\right)$

$$e^{\ln\left(\frac{v}{v_0}\right)} = e^{-\frac{k}{m}t}$$

$$\frac{v}{v_0} = e^{-\frac{k}{m}t}$$

raise both sides to power of e

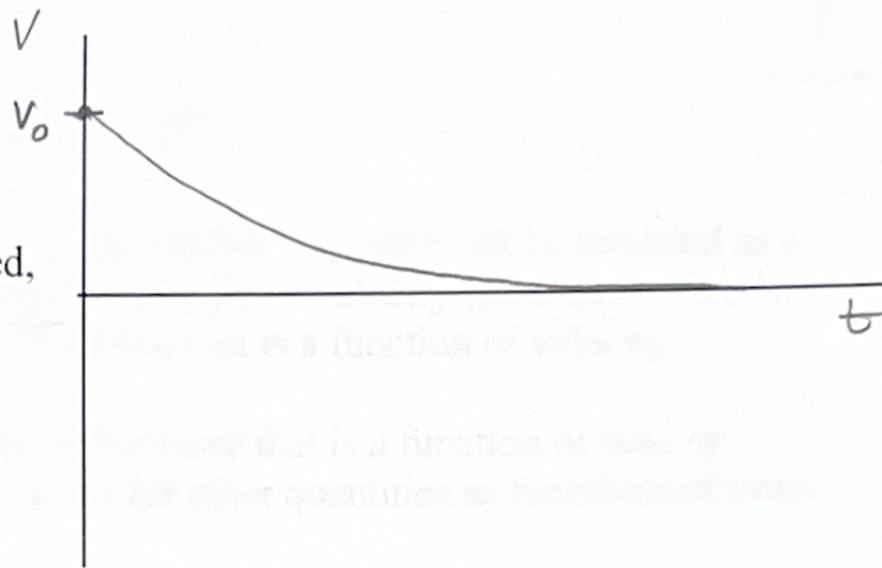
$$v = v_0 e^{-\frac{k}{m}t}$$

multiply both sides by v_0

b. Sketch the shape of the velocity vs. time graph, starting at $t = 0$.

Briefly explain what this graph reveals about the object's initial velocity, what is happening to its speed, and what its speed will be after a very long time elapses.

- initial velocity is +
- speed is decreasing
- after long Δt , $v = 0$



c. Find the block's position as a function of time

I have $v(t)$ and I want $x(t)$, so I will integrate.

$$v = v_0 e^{-k/mt}$$

$$\frac{dx}{dt} = v_0 e^{-\frac{k}{m}t}$$

$$dx = v_0 e^{-\frac{k}{m}t} dt$$

$$\int dx = \int v_0 e^{-\frac{k}{m}t} dt$$

u-Substitution

$$\text{let } u = -\frac{k}{m}t$$

$$\frac{du}{dt} = -\frac{k}{m}$$

$$dt = -\frac{m}{k} du$$

$$\int dx = v_0 \int e^u \left(-\frac{m}{k} du\right)$$

$$x = v_0 \left(-\frac{m}{k}\right) \int e^u du$$

$$x = -\frac{mv_0}{k} e^u + C$$

$$x = -\frac{mv_0}{k} e^{-\frac{k}{m}t} + C$$

$$\left[\begin{array}{l} 0 = -\frac{mv_0}{k} e^0 + C \\ \frac{mv_0}{k} = C \end{array} \right]$$

$$x = -\frac{mv_0}{k} e^{-\frac{k}{m}t} + \frac{mv_0}{k}$$

$$x = \frac{mv_0}{k} \left(1 - e^{-\frac{k}{m}t}\right)$$

d. Find the block's acceleration as a function of time

$$a = \frac{dv}{dt}$$

$$= \frac{d}{dt} \left(v_0 e^{-\frac{k}{m}t} \right) = v_0 e^{-\frac{k}{m}t} \left(-\frac{k}{m} \right) = -\left(\frac{kv_0}{m} \right) e^{-\frac{k}{m}t}$$

e. Find the horizontal force on the block as a function of time

$$F = ma$$

$$F = m \left(-\frac{kv_0}{m} \right) e^{-\frac{k}{m}t}$$

$$F = -kv_0 e^{-\frac{k}{m}t}$$

do you recognize this as $F = -kv$? 😊